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Note on Hecke operators and cohomology of groups

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Introduction.

In this note we study the action of Hecke operators on 1-dimensial cohomology group of the modular group $G=PSL_2(Z)$ with the coefficient module W, the even degree parts of the polynomial algebra $Z[x,y]$, or its reduction modulo a prime power ℓ, $W/\ell = (Z/\ell Z)[x,y]$. The cohomology group $H^n(G;W/\ell)$ is a module over $H^0(G;W/\ell)=(W/\ell)^G$, the invariants of W/ℓ. The ring $(W/\ell)^G$ is known by Dickson [1]. We notice the relation between the above module structure and the action of Hecke operators. Then we obtain some congruences for the eigenvalues of Hecke operators on modular forms.

Theorem. Let λ_ℓ be the eigen value of the Hecke operator T_ℓ in $M_k^0(G)$; the set of all cusp forms of weight k. Then

(1) $\lambda_5 \equiv 0 \bmod 5$ if $k \equiv 8, 10, 14 \bmod 20$

(2) $\lambda_7 \equiv 0 \bmod 7$ if $k \equiv 10, 14 \bmod 42$

(3) $\lambda_{11} \equiv 0 \bmod 11$ if $k \equiv 14 \bmod 110$.

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§ 1. Hecke operators and the Eichler-Shimura isomorphism.

Let $G=PSL_2(Z)$ be the modular group and $V=Z[x,y]$, $|x|=|y|=1$, be the polynomial algebra over Z. If we denote the positive even degree parts of V by W, G acts on W from the left. For any G-module E, the Eichler cohomology group $H_P^1(G;E)$ is defined to be the kernel of the restriction map $j^*: H^1(G;E) \to H^1(G_\infty;E)$, here G_∞ denotes the subgroup of G generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
From Lemma 2.2 in [9] or [3], in the case where $E=W^+ \otimes R$, ; the positive degree parts of $W \otimes R$, the above map j^* is epic. Therefore

(1.1) $H^1(G;W\otimes R) \doteq H_P{}^1(G;W\otimes R) \oplus H^1(G_\infty;W\otimes R)$.

Let us denote by $M_k(G)$ (resp. $M_k{}^0(G)$) the set of all automorphic (resp. cusp) forms of weght k with respect to G. Now we recall the actions of Hecke operators on cohomology groups and automorphic forms. Let α be an element of $M_2{}^+(Z) = \{A \in M_2(Z) \mid \det A > 0\}$. Then the double coset $G\alpha G$ decomposes into a disjoint union of finite number of left G cosets, $G\alpha G = \coprod_{i=1}^{d} G\alpha_i$. For $g \in G$, let $\alpha_i g = g_i \alpha_{i*}$ with some $1 \leq i* \leq d$ and some $g_i \in G$. Then for any G-module E, the Hecke operator $\widetilde{T}_\alpha$ on $H^1(G;E)$ is defined by

(1.2) $\widetilde{T}_\alpha u(g) = \sum_{i=1}^{d} \alpha_{i*} u(g_i)$ for $u \in Z^1(G;E)$.

The Hecke operator T_α on $M_k(G)$ is defined by

(1.3) $T_\alpha f(z) = \det\alpha^{k-1} \sum_{i=1}^{k} f(\alpha_i z) j(\alpha_i, z)^{-k}$ for $f \in M_k(G)$.

Here $j(\alpha_i, z) = c_i z + d_i$ for $\alpha_i = \begin{pmatrix} * & * \\ c_i & d_i \end{pmatrix}$.

Then there exists an R-linear isomorphism called the Eichler-Shimura isomorphism,

(1.4) $\varphi : M_{k+2}{}^0(G) \cong H_P{}^1(G; W^k \otimes R)$,

which commutes with Hecke operators (Shimura [10]).

In (1.4) W^k denotes the k-degree parts of W. Let E_k be the Eisenstein series $E_k(z) = \frac{1}{2\zeta(k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k}$

Proposition 1.5. The map φ in (1.4) is extendable to an R-linear isomorphism

$$\varphi : M^0_{k+2}(G) \oplus R \cdot E_{k+2} \cong H^1(G; W^k \otimes R),$$

which commutes with Hecke operators.

Proof. From the proof of Proposion 8.5 in Shimura [10], we can extend to $M^0_{k+2}(G) \oplus R \cdot E_{k+2}$ by defining

$$\varphi(f\ )(g) \quad = \sum_{j=0}^{k} x^j y^{k-j} \int_{z_0}^{g(z_0)} \mathrm{Re}(fz^j dz) \quad \text{for } f \in M_{k+2}(G),$$

and the φ commutes with Hecke operators. From (1.1), it is easily seen that the extended φ is an isomorphism, if

(1.6) the coefficient of x^k in $\varphi(E_{k+2})\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \int_i^{i+1} \mathrm{Re}(E_{k+2} z^k)\,dz \neq 0$.

We can prove that (1.6)>0 by direct computations for k=2 and 4, and by showing that $|B_k|/2k(k-1) > \sum_{n=1}^{\infty} \sigma_k(n)/e^{2\pi n}$ for $n \geq 6$, where B_k is the k-th Bernoulli number and $\sigma_k(n) = \sum_{d|n} d^k$.

Q.E.D.

We are indebted to S.Mizumoto for the proof of (1.6).

§2. Cohomology and congruence.

In this section we obtain some results about congruence properties of eigenvalues of Hecke operators on modular forms by studying the cohomology of G. The following two propositions are simple but fundamental. For a prime ℓ let W/ℓ be $W/\ell W$.

Proposition 2.1. If there is a $\lambda \in Z/mZ$ such that $\widetilde{T}_\alpha x = \lambda x$ for any $x \in H^1(G; W^k/\ell)$, then for any eigen value λ_α of T_α in $M_{k+2}(G)$, a conguence $\lambda_\alpha \equiv \lambda$ (modℓ) holds.

Proof. We have an exact sequence

$$H^1(G;W) \xrightarrow{\ell} H^1(G;W) \longrightarrow H^1(G;W/\ell).$$

From the assumption, we have $\widetilde{T}_\alpha x \equiv \lambda \widetilde{x}$ (mod ℓ $H^1(G;W)$) for any $x \in H^1(G;W)$. Hence the proposition follows from Proposition 1.5.

Q.E.D.

By the cup product, $H^1(G;W/\ell)$ is an $H^0(G;W/\ell) = (W/\ell)^G$ module. The action of $(W/\ell)^G$ is defined by $(wu)(g) = w \cdot u(g)$ for $w \in (W/\ell)^G$, $u \in Z^1(G;W/\ell)$ and $g \in G$.

Proposition 2.2. Let $w \in (W/\ell)^G$ and $\alpha \in M_2^+(Z)$. If there is a $\lambda \in Z/\ell Z$ such that $\alpha w = \lambda w$, then $\widetilde{T}_\alpha wu = \lambda w \widetilde{T}_\alpha u$ for any $u \in H^1(G;W/\ell)$.

Proof. By the definition (1.2),

$$\begin{aligned} \widetilde{T}_\alpha wu(g) &= \sum \alpha_{i*}(wu)(g_i) \\ &= \sum \alpha_{i*} w\, \alpha_{i*} u(g_i). \end{aligned}$$

Since $\alpha_{i*} = g_i^{-1} \alpha g \in G\alpha G$, $\alpha_{i*} w = \lambda w$ holds from the assumption. Terefore

$$\widetilde{T}_\alpha wu(g) = \lambda w \sum \alpha_i u(g_i) = \lambda w \widetilde{T}_\alpha u(g). \qquad \text{Q.E.D.}$$

Next we consider the invariant $(W/\ell)^G$ for a prime number ℓ. We define two elements E_1 and E_2 in $V=Z[x,y]$ by

$$E_1=xy^\ell-x^\ell y \quad (\text{ when } \ell=2,\ E_1=(xy^2-x^2y)^2) \quad \text{and}$$
$$E_2=x^{\ell(\ell-1)}+x^{(\ell-1)(\ell-1)}y^{(\ell-1)}+\cdots+y^{\ell(\ell-1)}.$$

Then the classical result of Dickson [1], says that $(W/\ell)^G=(Z/\ell Z)[E_1,E_2]$ and moreover W/ℓ is a free $Z/\ell Z[E_1,E_2]$-module.

It is wellknown that $G=PSL_2(Z)$ is the free product $Z/2Z * Z/3Z$. Here $Z/2Z$(resp. $Z/3Z$) is generated by $\tau=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ (resp. $\sigma=\begin{pmatrix}1&1\\-1&0\end{pmatrix}$)[7]. Therefore, about the classifying space we have

$$BG \simeq BZ/2Z \vee BZ/3Z$$

where $\vee$ denotes the one point union. For any G-module E, we have the Mayer-Vietories exact sequence

$$(2.3)\quad E^{Z/2Z}\oplus E^{Z/3Z}\to E\to H^1(G;E)\xrightarrow{i^*}H^1(Z/2Z;E)\oplus H^1(Z/3Z)\to 0$$

Proposition 2.4. The $Z/\ell Z[E_1,E_2]$-module $H^1(G;W/\ell)$ is generated by generators of degree equal or less than ℓ^2-1 (6 for $\ell=2$).

Proof. The free $Z/\ell Z[E_1,E_2]$-module W/ℓ is generated by the degree equal or less than

$$|E_1|+|E_2|-2=\ell^2-1 \quad (=6 \text{ for } \ell=2).$$

Hence so is the quotient module $(W/\ell)/((W/\ell)^{Z/2Z}+(W/\ell)^{Z/3Z})$. We also prove $H^1(Z/2;W/2)$ (resp. $H^1(Z/3;W/3)$) is generated by elements degree ≤ 6 (resp. 8) from the explicit computation of the cohomology. (see[9]). Q.E.D.

Theorem 2.5. Assume $\alpha E_i = E_i$ (resp. $\alpha E_i = \mu_i E_i$ for some $\mu_i \in Z/\ell Z$) and there is a $\lambda \in Z/\ell Z$ such that $\widetilde{T}_\alpha x = \lambda x$ (resp. $\widetilde{T}_\alpha x = 0$) for any $x \in H^1(G;W^k/\ell)$ with $0 \leq k \leq \ell^2-1$ (when $\ell=2$, $0 \leq k \leq 6$). Then for any eigen values λ_α of T_α in M_{K+2}, with any $K \geq 0$, the congruence $\lambda_\alpha = \lambda \mod \ell$ (resp. $\lambda_\alpha = 0 \mod \ell$) holds.

Proof. Any element $f \in H^1(G;W/\ell)$ can be written as $f = \sum a_i f_i$ here $a_i \in Z/\ell Z[E_1, E_2]$ and $|f_i| \leq \ell^2-1$. . Then from the assumption and Proposition 2.2,

$$\widetilde{T}_\alpha f = \sum a_i \widetilde{T}_\alpha f_i = \sum a_i \lambda f_i = \lambda f.$$

So the assersion follows from Propositiion 2.1. Q.E.D.

Let p be a prime number. Let us write $T_\alpha = T_p$ for $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. It is immediate that

$$\alpha E_i = E_i \text{ if } p=1 \mod \ell \text{ or if } p=-1 \mod \ell \text{ and } i=2$$

$$\alpha E_1 = -E_1 \text{ if } p=-1 \mod \ell .$$

Therefore if $p = \pm 1 \mod \ell$ and $\widetilde{T}_p x = (1+p)x$ in $H^1(G;W^k/\ell)$ for $k \leq \ell^2-1$, then $\lambda_p = (1+p) \mod \ell$ hold in $M_s(G)$ for all positive degree weigt s. It is known by Hatada [4] (Heberland [3], Papier [5], [6]) that for $\ell \leq 7$ the above congruence hold. However it seems difficult that one check $\widetilde{T}_p x = (1+p)x$ for $\ell \geq 5$ now.

Recall $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $G_\infty = \langle \gamma \rangle$ and $j : G_\infty \hookrightarrow G$ is the inclusion map. The cohomology of G_∞ is easily computed

Lemma 2.6. $H^1(G_\infty;V/\ell) \simeq (V/\ell)/\mathrm{Im}(\gamma-1)$

$$\simeq Z/\ell Z[V] \otimes (Z/\ell\{1,x,..,x^{\ell-2}\} \oplus Z/\ell Z[y]\{x^{\ell-1}\})$$

where $V = x^\ell - xy^{\ell-1}$ and $Z/\ell Z\{a,b,..\}$ is the Z/ℓ-module generated by a,b,...

It is easily seen that $GaG = \prod Ga_i$, $a_i = \begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix}$ for $0 \leq i \leq p-1$, $a_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

Lemma 2.7. $j^*(T_p\varphi) = (a_0 + pa_p) j^*\varphi$ in $H^1(G_\infty; W/\ell)$ for $\varphi \in H^1(G; W/\ell)$.

Proof. Direct computation shows that $a_j\gamma = 1 \cdot a_{j+1}$ for $0 \leq j \leq p-2$, $a_{p-1}\gamma = \gamma a_0$ and $a_p\gamma = \gamma^p a_p$. Therefore we have

$$\tilde{T}_p\varphi(\gamma) = \sum_{j=0}^{p-2} a_{j+1}\varphi(1) + a_0\varphi(\gamma) + a_p\varphi(\gamma^p)$$
$$= a_0\varphi(\gamma) + a_p(1+\gamma+\ldots+\gamma^{p-1})\varphi(\gamma)$$
$$= a_0\varphi(\gamma) + pa_p\varphi(\gamma) + \sum_{i=0}^{p-1}(\gamma^{p^i} - 1)\, a_p\varphi(\gamma).$$

since $a_p\gamma^j = \gamma^{p^j} a_p$. Q.E.D.

Corollary 2.8. (i) If $p = \pm 1 \bmod \ell$, then $j^*(T_p\varphi) = (1+p) j^*(\varphi)$.
(ii) If $p = 0 \bmod \ell$, then $j^*(T_p\varphi) = j^*(\varphi)$ for $j^*(\varphi) = v^s x^i$ and $j^*(T_p\varphi) = 0$ for $j^*(\varphi) \in$ Ideal y.

Theorem 2.9. Let $\ell - 1 < k < \ell^2 - 1$ and $M_{k+2}^0(G) = 0$. Then the eigenvalue $\lambda_p = 0 \bmod \ell$ for the Hecke action T_p in $M_{s+2}^0(G)$ where $s = k \bmod \ell(\ell-1)$.

Proof. Consider the maps

$$H^1(G;W) \xrightarrow{r} H^1(G;W/\ell) \xrightarrow{i^*} H^1(G_\infty; W/\ell)/(\text{Ideal } y).$$

From corollary 2.8, $H^1(G; W^s/\ell)$ decomposes as $\mathrm{Ker}\, i^* \oplus E$ with $E \simeq Z/\ell$. Let us take $H^1(G;W) = K \oplus E$ with $r(K) = \mathrm{Ker}\, i^*$ and $r(E) = E'$. Hence $E \simeq Z$ because for the Eisenseries, the Hecke action operates $T_\ell(E_{s+2}) = E_{s+2}$ and from Proposition 1.5, $E \otimes R \simeq R$. Since $K \otimes R \simeq H_p^1(G; W^s) \otimes R$ and E is torsion free, K is closed under T_ℓ.

By the assumption $M_{k+2}^0(G) = 0$, K^k is torsion. The ℓ-torsion in $H^1(G;W)$ is isomorphic to $H^0(G; W^+/\ell) \simeq Z/\ell[E_1, E_2]^+$ from the exact sequence

$$H^0(G;W) \xrightarrow{r} H^0(G;W/\ell) \xrightarrow{\delta} H^1(G;W) \xrightarrow{\ell} H^1(G;W)$$

and $H^0(G;W)=W^0$. Therefore $K^k = Z/\ell\{E_1{}^m\}$ where we only need to consider the ℓ-torsion since the lowest dimensional ℓ^2-torsion element is E_1^ℓ.

Let $f \in \mathrm{Ker}\, i^*$. Since $\ell-1<k<\ell^2-1$ and $s \equiv k \mod \ell(\ell-1)$. we can take

$$f = E_2{}^r f_1 + E_1 g, \qquad |f_1| = k,\ r>1.$$

Note that $\alpha E_1 = x^\ell(\ell y) - (\ell y)^\ell x = 0$ in W/ℓ and Proposition 2.2, $\tilde{T}_\ell E_1 g = (\alpha E_1)\tilde{T}_\ell g = 0$. Since $f_1 = \lambda r\delta(E_1{}^m)$, $\lambda \in Z/\ell$,

$$E_2{}^r f_1 = E_2^r r\delta(E_1{}^m) = E_1{}^m E_2{}^{r-1} r\delta(E_2).$$

Hence $\tilde{T}_\ell(E_2{}^s f_1)=0$. Therefore $\tilde{T}_\ell(\mathrm{Ker}\, i^*)=0$. Since K is closed under $\tilde{T}_\ell$, $\tilde{T}_\ell(K)=0 \mod \ell(K)$. Since $K \otimes R \simeq H_p{}^1(G;W)\otimes R$, we have the theorem. Q.E.D.

The fact $M_{s+2}{}^0(G)=0$ for $s+2 \leq 14$ and $\neq 12$ implies the theorem in the introduction.

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